Swapskills Pte Ltd

The Greeks and Pricing

Options

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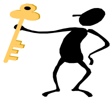


Options are used for protecting ourselves in times of unfavourable price movements in an underlying asset. There's a cost associated with purchasing the option which is payable upfront and is called the premium. A call option makes a payout, when the price rises above the strike price and the put option makes a payout when the price falls below the strike price. If the price trends in a favourable direction, the option will expire worthless but the asset can be purchased/ sold directly from/to the market at a better price than under the option provision.

An option's cost consists of an intrinsic value (payoff to the option) and a time value (the cost of holding onto the option for a longer period when the price of the underlying can trend up or down). We'll now take a more detailed look at pricing up an option to a precise dollar value, taking into account all of the factors that contribute to a change in its price, which leads us to a question—

What contributes to the price of an option?

Inside this issue:

Points of interest

- **Isolate the risks associated with holding an option based on factors that drive its price**
- **Options can be evaluated in continuous or discrete time**
- **The Black Scholes mathematical model to derive formulae for option price sensitivities to different factors**

What are the factors contributing to option's price? - An intuitive discussion!

Asset Price (S):

As the asset price increases

Payout to a call increases

Call value increases

Payout to a put option decreases

Put value decreases

Volatility of asset returns (σ):

As the volatility of the underlying increases

Demand for options increases

Put value increases

Call value increases

Risk Free Interest Rate (r):

This comes from the BSM* model.

As interest rates increase

Call value increases

Put value decreases

Time to expiration (T):

As expiry date draws near

Less time for asset price volatility

Call value decreases

Put value decreases*

[*For deep in the money puts, prices may increase]

Strike Price (X):

As Strike increases,

Less likely will a call make a payout

Call value decreases

Higher payoff from put expected

Put value increases

There are 2 ways to think if the price of an option will increase or decrease

A) Considering the fundamentals—Does the intrinsic value change based on the change in value of the factor

B) Supply and Demand: Will the change in value of the factor drive up supply / demand for options? If demand increases price will increase!

*BSM—Black Scholes Merton

Pricing models

Given that we now know the factors that affect the cost of the option, we are looking for a relation to describe exactly how the option cost varies as a the function of factors. There are two such models which are most commonly used.

Binomial Option pricing model

This model values an option as the probabilistically weighted average of the value of the option based on 2 different values of the underlying one ***discrete*** time interval later. It is enough to know the beginning asset value, the size of the 2 possible price movements and their probability to price up the option.

Black Scholes Merton (BSM) Model

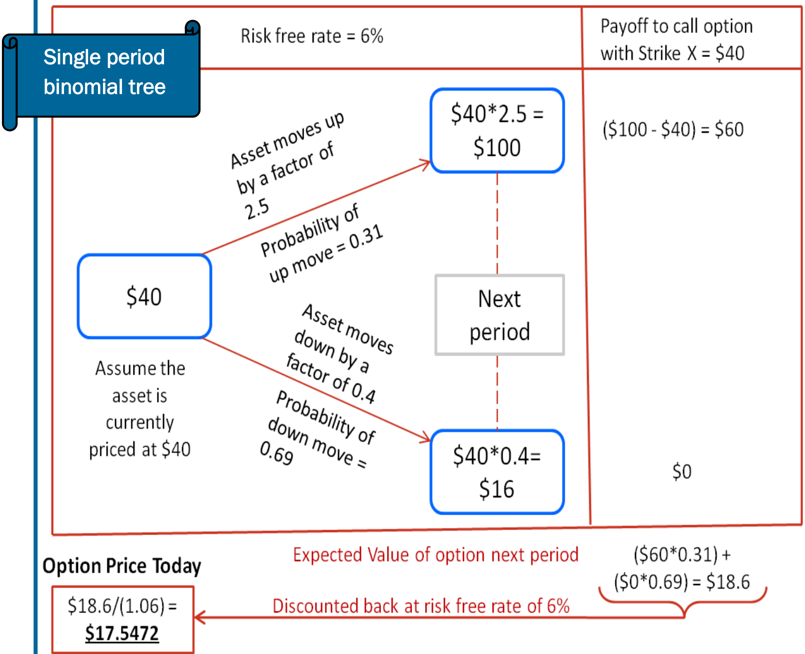
This model values an option in ***continuous*** time. It assumes a log-normal distribution of the price of the underlying, a constant continuous risk free rate of interest, a constant volatility of underlying, frictionless markets, no dividends / coupons on the underlying and European style options.

A Bit of History

Merton and Scholes received the 1997 Nobel Prize in Economics (The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel) for their work. Though ineligible for the prize because of his death in 1995, Black was mentioned as a contributor by the Swedish academy.

Binomial Option Pricing model

This is a simple model where we assume that some value will change to one of any 2 possible values (binomial). Here's how it works—



The asset's prices are shown in the blue boxes. The asset can either move up by a factor (U = 2.5) or move down by a factor (D = 0.4) over the next period. Let's assume that the up move happens with a probability of P(U) = 0.31, and the down move happens with a probability of P(D) = 0.69. The risk free rate of interest is Rf = 6%

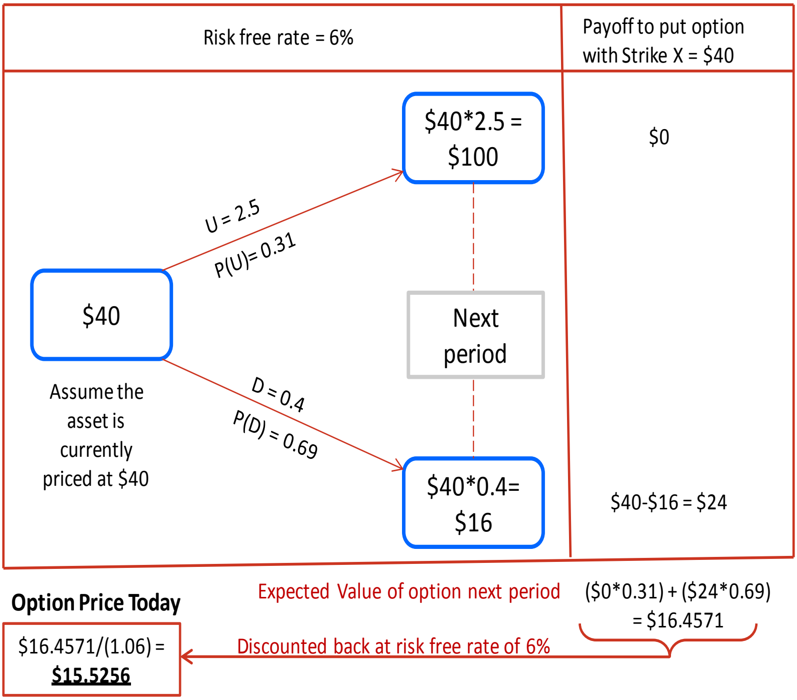
If an up move happens, the value to the call option is $60 and if the down move happens the call expires worthless. Hence the value to the call at the end of the next period is a probabilistic weighted average and equals $18.6. We obtain the value of the call today by discounting the call value from the next period at the risk free rate of interest.

Note: D = 1/U
Also note: P(U) = (1+Rf-D)/(U-D) and P(D) = 1—P(U). P(U) and P(D) are called pseudo -risk neutral probabilities and assume investor risk-neutrality.

An example for you on valuing a put option!

Take a look at the asset in the previous page. Assuming similar probabilities of up and down moves and similar price movements of the asset, what would be the value had it been a put option?

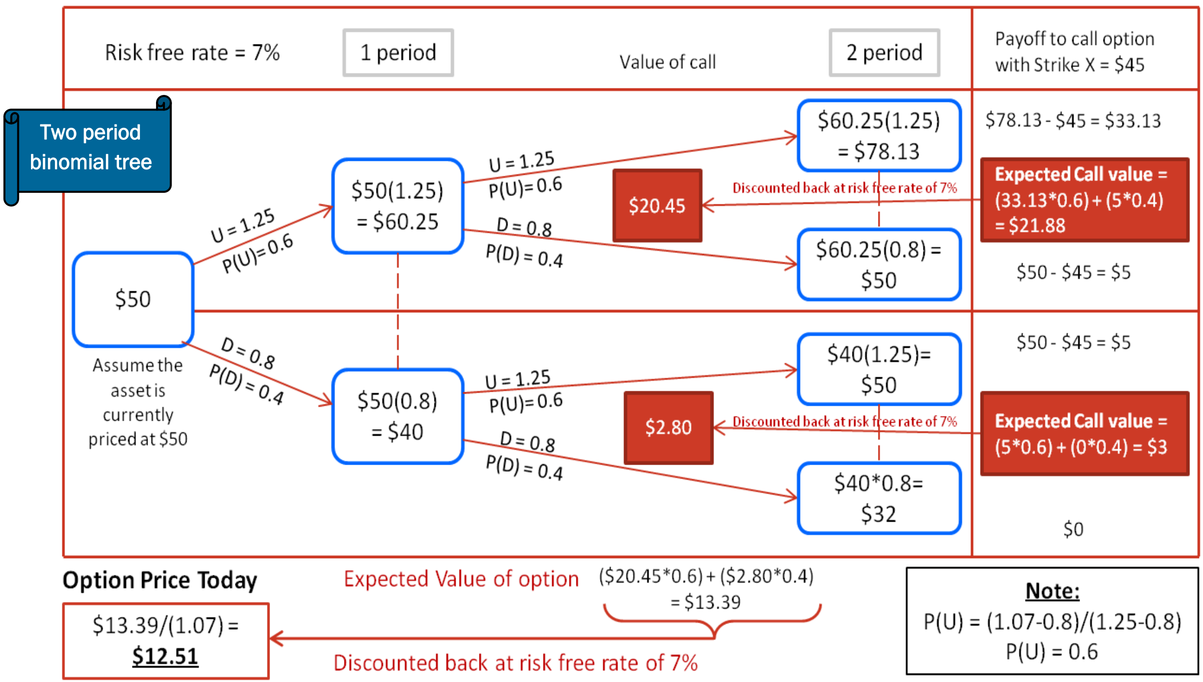
Answer: It's exactly similar to valuing the call, except for the payoffs.



Alternatively we could have evaluated the value of the put option using put-call parity as we know all the other variables. This yields nearly the same result!

Po	=	Co	-	S	+	X/(1+Rf)^1
15.2830		17.54717		40		37.7358

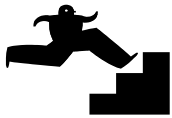
Two period Binomial Model



Two Period Binomial Model explained

A 2 period binomial model can be used to evaluate an option that will expire periods later.

Using the model is much like using a single period model except that we will evaluate the value of an option one period at a time by discounting its expected payoff to a present value.



The steps are simple—

A) Compute the probability of uptick and downtick from the tick size and the risk free rate.

B) Compute the value of the asset given all combinations of upticks and downticks. This leaves us 4 possibilities for a forecasted asset value after 2 periods.

C) Evaluate the payoff for a call option 2 periods later for each asset price move.

D) Calculate the expected payoff as the probabilistic weighted average and discount it back one period to obtain the value of the call in one period for the upper and lower halves of the binomial tree.

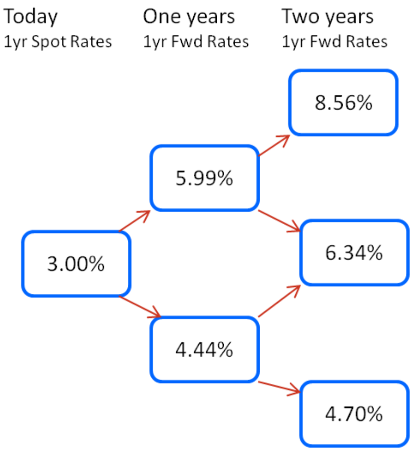
E) Repeat the process of taking the probabilistic weighted average and discounting the expected payoff for another period to compute today's option value.

Question: What would change in the 2 period binomial model if it were a put option instead?

Answer: Only the payoffs would change! Compute the payoffs to a put option for a given value of asset price at the end of the second period and discount the expected value of the payouts for 2 periods in order to derive the value of a put.

Binomial Option pricing for Bond options

Interest Rate Trees:
Similar to how an asset price can move up or down over the next period, so can one period forward interest rates. This is the basis for evaluating bond options and interest rate options using a binomial model, and the binomial tree that can be created for interest rates is called an interest rate tree. Let's take a look at an example of this—



How does this differ from the previous tree we saw for asset price volatility—

A) The rates are one period forward rates, i.e. the one period lending rates one year from now can move to one of 2 values. (5.99% or 4.44% in this case)

B) The size of the uptick and downtick aren't pre-determined factors. They are based on an inherent ***interest rate volatility assumption*** that is made before constructing the tree. A change in the assumption will change the rates in every node of the tree.

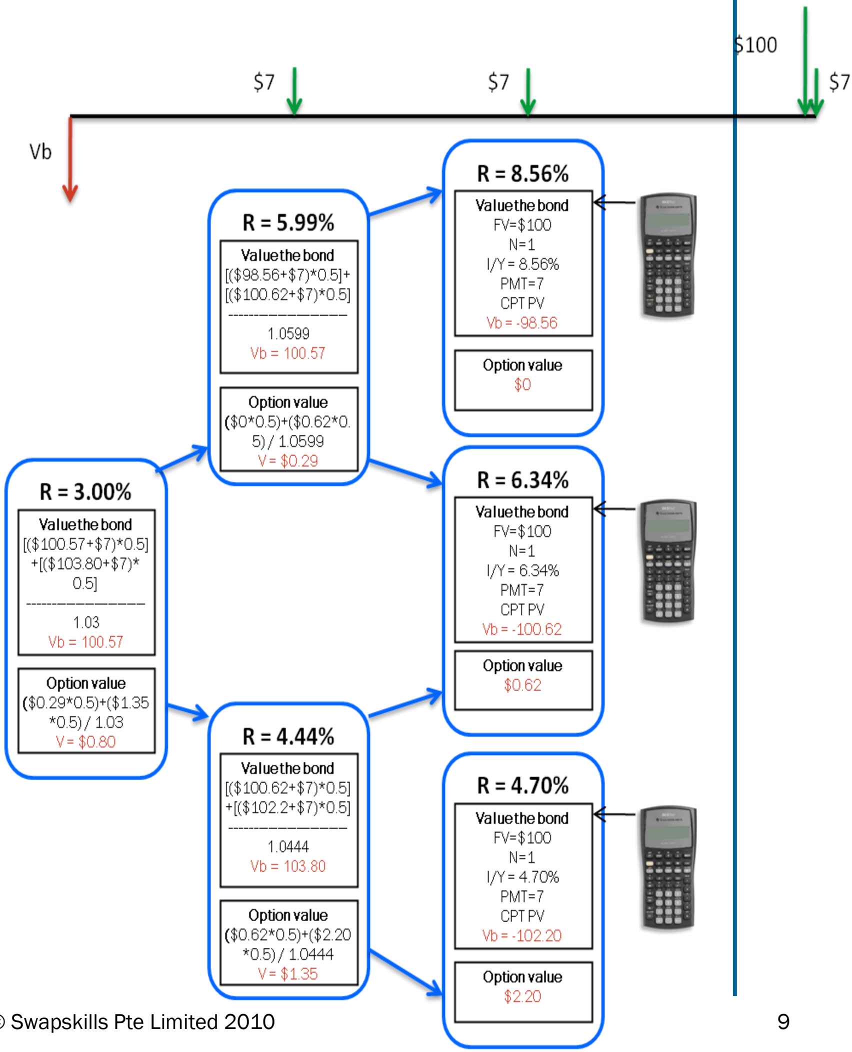
C) There is no single risk-free rate. The cash flows to a bond or an interest rate option will be discounted over any period using the forward rates derived from the interest rate tree.

D) The probability of uptick and downtick are fixed at 0.5 and are independent of tick size and risk free interest rates.

Having an binomial interest rate tree is important in bond option valuation as the bond's price vary based on interest rate movements which will have an effect on what value is received to the option!

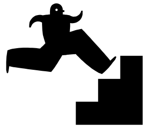
Two period binomial option pricing model for bond options

You want to evaluate a European call option with 2 years to expiration and a ***strike price of $100***. The underlying bond is a $100 par value, 7% annual coupon bond with three years to maturity. Compute the value of the option today!



Two period binomial option pricing model for bond options explained

To value the European call option we will have to determine the value to the call at each node. The value of the call option is determined by the bond price at each node. We hence follow the following steps to value the call—



1. Value the bond at every node for year 2, by discounting the final coupon and principal payment for one period using the one period forward rate. The steps are shown on the calculator.

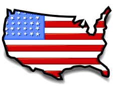
2. Value the bond at every node for year 1 by discounting the expected value of the bond and the coupon at the one year forward rates for one period. This method is called backward induction

3. Value the bond at the root node at year 0 using backward induction.

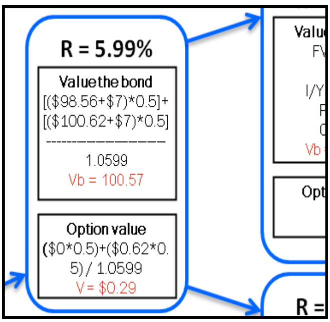
4. The intrinsic value of the call option at year two is the excess of the bond value over the strike price, or zero if the bond price is less than the strike price.

5. Discount the expected value to the call using the one period forward rates to obtain the value of the call options for the year 1 nodes.

6. Repeat the process to obtain the value of the call option today!



What if it had been an American call option?

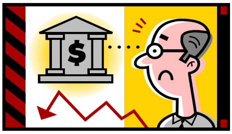


Take a look at the top node in year 1 as shown. We'd evaluated the value of the call option at the node as $0.29. However the intrinsic value of the call if it were exercisable is ($100.57—$100) = $0.57.

We know that an American option can be exercised prior to expiry, so in this case we could buy the option at 0.29 and immediately sell for 0.57 and make a small arbitrage profit.

However if the option is correctly priced, arbitrage must not be possible. Hence we need to replace the option value at each node with the maximum of its intrinsic value or present value of future payoffs!

Options on Interest Rates

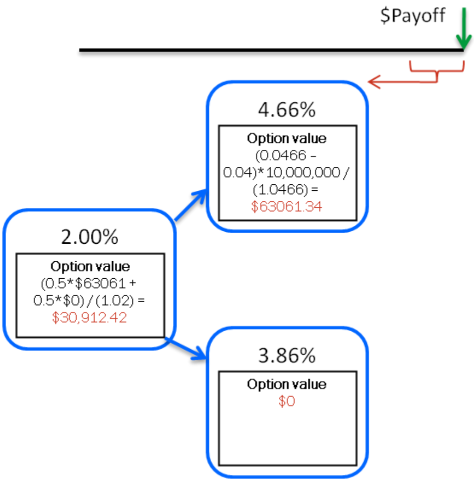


An interest rate option is one in which the underlying is a rate. The payoff is given by how much the underlying rate exceeds the strike rate for a unit of principal. The notional is agreed when entering into the options contract.

The most interesting point about interest rate options is that they ***pay in arrears***, i.e. they make a payment one period after the expiration of the option.

Valuing an Interest rate call option

Given the following one year forward rate binomial-tree, find the price of an interest rate call option with a strike rate of 4%., which expires in one year. The notional agreed is $10,000,000.



Answer:
The interest rate call option will make a payoff one period after the first year as shown in the case of the node above and expire worthless in the case of the node below.

The option value at any node is the present value of the future payoffs to be received from the call option.

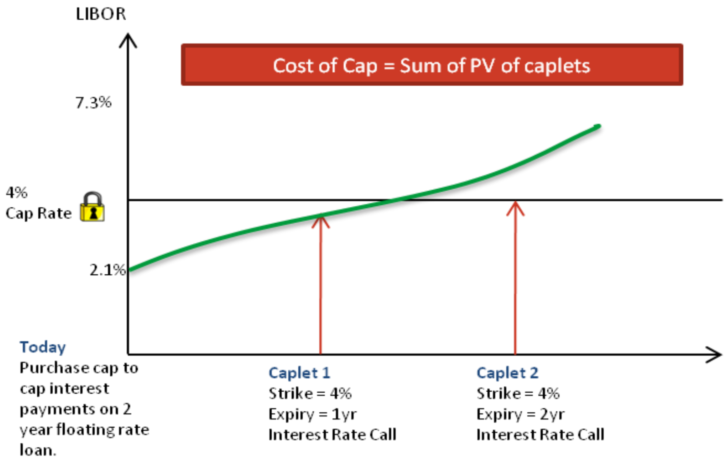
Once we have computed the option value at both nodes as shown, we can discount the expected payout (probabilistic weighted average) at the one year spot rate of 2% for one year to obtain the value of the option today!

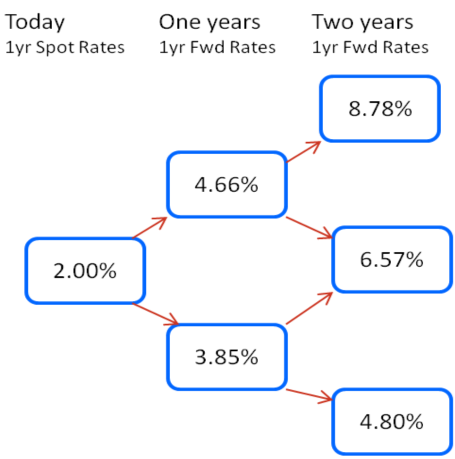
What if it were an option for a 2 year period?

Really straightforward! In this case, we would have simply used a 2 period binomial model like we have seen before. That would give us 4 possible one period forward rates for the second year and we would work our way one period at a time using backward induction like we have seen before.

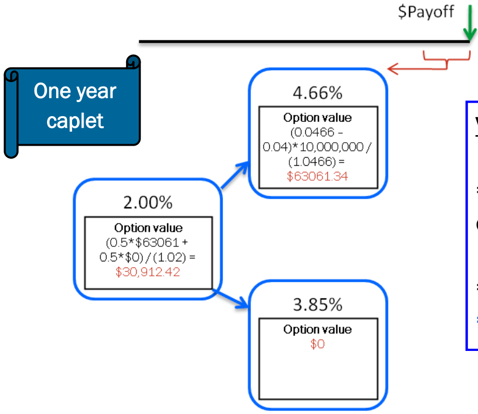
Valuing an Interest Rate Cap

You are borrowing at LIBOR over the next 2 years and are afraid that if LIBOR were to rise above 4% your interest payments might be too expensive to service. You hence decide to buy a 2 year cap with a strike of 4% and a notional equal to the notional of the loan which is 10,000,000. What would be a fair value to pay for this cap given the interest rate tree below?





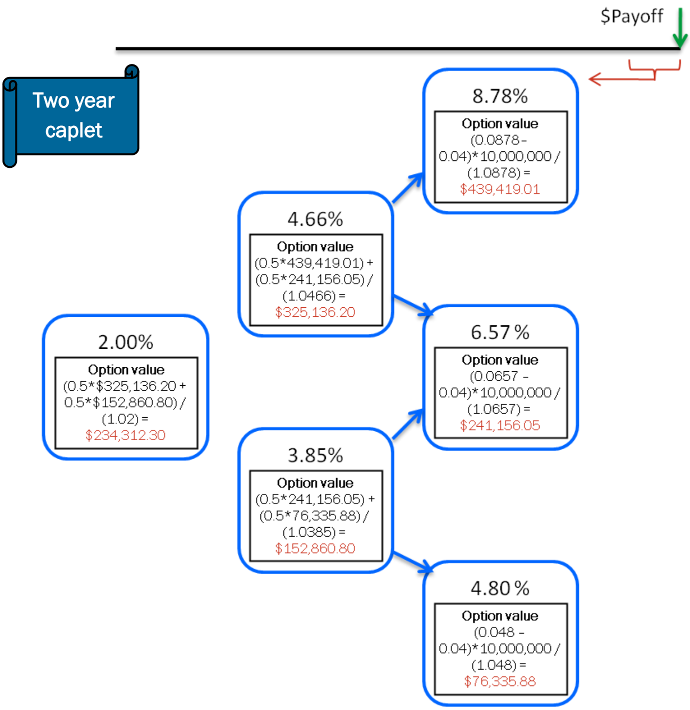
<u>Answer:</u> The interest rate cap is nothing but simultaneously purchasing 2 interest rate call options with different expiry dates and the same strike rate. The value of the cap is hence the value of the 2 options called “caplets”. We will use a one period and 2 period binomial model for valuing the 2 caplets. Take a look—



Value of the 2 year cap

= Value of 1 year caplet + Value of 2 year caplet

= $30,912.42 + $234.312.30
= $265,224.72



Black Scholes Merton Model

The Black, Scholes and Merton (BSM) model for option pricing assumes that the price of heavily traded assets follow a geometric Brownian motion with constant drift and volatility. It was developed in 1973 by Fisher Black, Robert Merton and Myron Scholes and is still the most widely used model to price European call options today.

What factors get priced in?

When applied to stock options the model incorporates the following factors into the price of an option—

The option's strike price 	the constant price variation of the stock 	the time value of money 	the time to the option's expiry 

The Assumptions behind the model

The price of the underlying asset follows a lognormal distribution—*(ie the log of asset prices follows a normal distribution)*

The continuous risk free rate is a constant and is known—Hence the BSM Model is not useful for pricing bond options and interest rate options as interest rate volatility must be factored in.

The underlying asset has no cash flows like dividend / coupon payments—There are however variations of the model for assets which make dividend payouts.

The volatility of the underlying asset is constant and known—*This doesn't hold often and BSM model is not useful in these cases.*

Markets are frictionless—There are no taxes, no transaction costs, and no restrictions on short sales or the use of short-sale proceeds.

Options are European options which can only be exercised at maturity—American options aren't priced correctly using BSM. They are rather priced using Binomial option pricing model.

The Formula

$$C_0 = [S_0.N(d_1)] - [Xe^{-R_f^c T}.N(d2)]$$

$$d_1 = \frac{\ln(S_0/X) + \left[R_f^c + (0.5 \times \sigma^2)\right] \times T}{\sigma \times \sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/X) + \left[R_f^c - (0.5 \times \sigma^2)\right] \times T}{\sigma \times \sqrt{T}} = d_1 - \left(\sigma\sqrt{T}\right)$$

From Put—Call parity

$$P_0 = C_0 - S_0 + \left[Xe^{-R_f^c T}\right]$$

$$P_0 = [S_0.(N(d_1) - 1)] - [Xe^{-R_f^c T}.(N(d2) - 1)]$$

Where

$T = $ *time to maturity* (*as* % *of* 365 *day year*) $= \tau - t$

$t = $ *time elapsed since beginning of option contract*

$S_0 = $ *asset price* $X = $ *Exercise price* $\tau = $ *Tenor of opinion*

$R_f^c = $ *Continuously compounded Risk Free rate*

$\sigma = $ *volatility of continously compounded returns on the stock*

$N(\cdot) = $ *cumulative normal probability function*

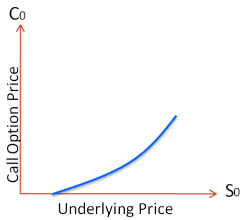
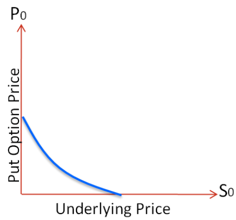
The Interpretation

The basis for this formula is complex but its application is rather simple. The Black Scholes is a mathematical model that gives a precise value of the price of calls and puts in terms of the factors which affect their price.

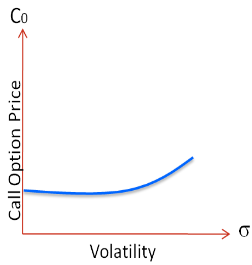
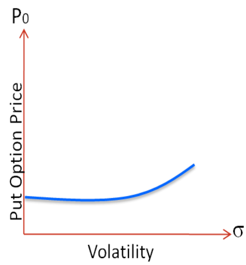
In order to isolate the effect each factor has on the price of the option, we differentiate the Black-Scholes equation "partially" with respect to each factor variable. We will thus be able to derive separate equations to model the movement of the price of the option with respect to the factor in question keeping all other variables constant!

These partial differentials are known as ***the Greeks***.

The Greeks

Delta (Δ)	
Formulae	$$\Delta\,(for\ calls) = \frac{\partial C_0}{\partial S_0} = N(d_1) > 0$$ $$\Delta\,(for\ puts) = \frac{\partial P_0}{\partial S_0} = N(d_1) - 1 < 0; as\ N(d_1) < 1$$
What it means	It is the relationship between asset price and option price. For a call option as asset price increases, option value increases, hence delta is positive. For a put option as asset price increases, option value decreases, hence delta is negative.
Plots	 
Interpretation	As the (N(d1)) increases, the price of a call increases more per unit increase in asset price, and hence delta increases. For put options however as N(d1) increases, 1-N(d1) decreases in magnitude and delta decreases. Hence delta for options is a function of N(d1).
Example	The delta for a call option is 0.533. The returns for an asset over the past period (last 10 mins of trading) has been 8% and the original price of the asset was $4. The option was priced at $12 initially. Use an approximation to find the new option value. **Answer:** Since the change in asset price is small, we can by linear approximation that ΔC = (delta)ΔS. Hence: (X–$12) = (0.533)($4(1.08) - $4). Solving, we get X = new option value = $12.1706

Vega(v)

Formulae	$$v\ (calls\ and\ puts) = \frac{\partial C_0}{\partial \sigma} = S_0.N'(d_1)\sqrt{T}$$
What it means	It is the relationship between volatility of returns on the underlying asset and the option price. When volatility is higher, all other factors constant, both puts and calls are more valuable. So Vega is positive for both calls and puts.
Plots	 
Interpretation	As time to maturity draws near (T decreases), period of volatility decreases and vega falls. If the price of the asset changes (S), vega changes. Further even if the expected volatility (N(d1)) changes, the vega changes. The way to interpret vega is the dollar price change in the value of an option corresponding to a 1% price change
Example	When the implied volatility increased over the past period by 3%, the price of the option increased by $6. Compute the vega for the option. **Answer:** Vega = Change in call value / Change in volatility Vega = 6/3 = 2

Rho(ρ)

Formulae

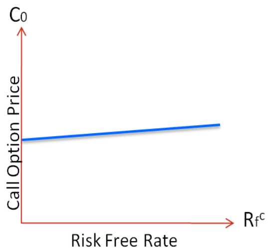
$$\rho\ (for\ calls) = XTe^{-R_f^cT}N(d2)$$

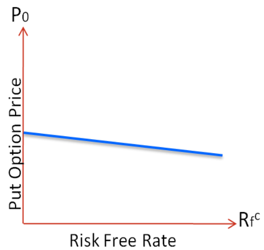
$$\rho\ (for\ puts) = XTe^{-R_f^cT}(N(d2) - 1)$$

What it means

It is the sensitivity of the option price to changes in the risk free rate. Generally the value of a call or a put doesn't vary much with changes in the risk free rate and hence rho isn't the most important sensitivity measure. We know that N() is a probability and is less than 1. All other terms being positive we notice that rho for calls is positive and rho for puts is negative. This means the value of a call rises and the value of a put falls with the rise in interest rates.

Plots





Interpretation

If X the strike price increases, rho increases in magnitude for both puts and calls (sign of rho for puts is negative), option value changes more per unit change in risk free rate. As time to maturity increases, the magnitude of rho increases as there is a greater period for interest rate volatility.

Example

The sensitivity of a call option to risk free rate is 0.85 and the cumulative normal probability N(d2) is 0.4. How much will the value of an otherwise identical put option change for a 2% rise in the risk free rate?

Answer:

ρ(put)/ρ(call) = - (1-N(d2))/N(d2) = -(0.6)/(0.4) = -1.5

ρ(put) = -1.5 x ρ(call) = -1.5 x 0.85 = -1.275

Hence if interest rate increased 2%, the value of a put would decrease by 2.55%.

Theta(Θ)

Formulae

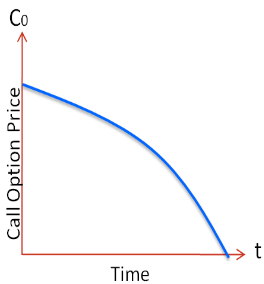
$$\theta\ (for\ calls) = \frac{\partial C_0}{\partial t} = -\frac{SN'(d1)\sigma}{2\sqrt{T}} - R_f^c X e^{-R_f^c T} N(d2)$$

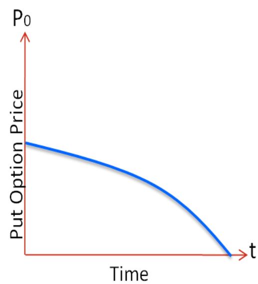
$$\theta\ (for\ puts) = \frac{\partial P_0}{\partial t} = -\frac{SN'(d1)\sigma}{2\sqrt{T}} - R_f^c X e^{-R_f^c T} (N(d2) - 1)$$

What it means

Theta is also called "Time Decay". It measures the sensitivity of the option price to the **passage of time**. As time to maturity draws near all else equal, the period of volatility decreases and the option value decreases. This relation holds for both calls and puts. However for deep in the money puts the value of a put option may increase as time passes owing to potential asset price volatility over the remainder time eroding away the value of the put.

Plots





Interpretation

As time (t) increases, time to maturity (T) decreases and theta (from the formula) decreases. Hence the value of the option decreases for a unit in time elapsed. What is important to understand here is that the relation between time (t) and option value is negative and the relation between time to maturity (T) and option value is positive.

Example

In 5 seconds of trading the value of a call option fell by \$1.3. Calculate the theta of the call option.

Answer:

Since the time interval is small, we can approximately say– ΔC = Θ(Δt). Hence, Θ = 1.3/5 = 0.26.

Exercise price sensitivity

What it means

The sensitivity to exercise price of an option is not a part of the Greeks, but worth studying at this juncture as it constitutes a factor of the Black Scholes pricing model.

From Black Scholes we have—

$$C_0 = [S_0.N(d_1)] - [Xe^{-R_f^L T}.N(d2)]$$

$$P_0 = [S_0.(N(d_1) - 1)] - [Xe^{-R_f^L T}.(N(d2) - 1)]$$

We can see that as exercise price X increases, the value of the call decreases all others factors constant and the value of the put increases. Hence exercise price sensi-

Interpretation

As the strike price increases, less likely will the call make a payoff, and hence its value decreases. Similarly if the strike price rises too high, the greater the probability of payout to the put and hence put value increases.

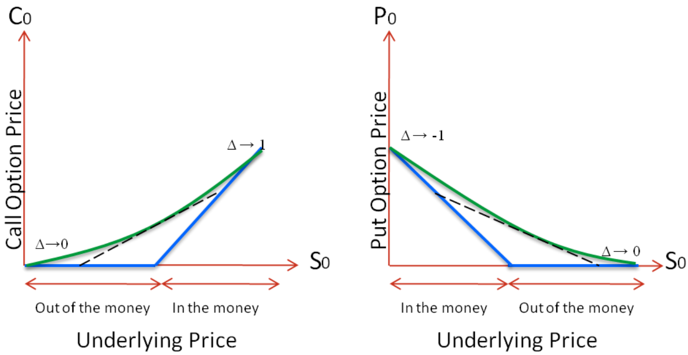
Remember this!

Snapshot summary

	Sensitivity of option price	For Calls	For Puts
Delta	Asset Price (S)	>0	<0
Vega	Volatility (σ)	>0	>0
Rho	Risk-free rate (r)	>0	<0
Theta	Time elapsed (t)	<0	<0
	Exercise price (X)	<0	>0

More about Delta

We now know that options have both intrinsic value and a time value. The intrinsic value (blue curve) and the option value prior to expiration (green curve) (which includes for the time value of the option) for a call option and a put option are as shown below—



The slope of this curve = (change in option price) / (change in price of underlying) = Delta!

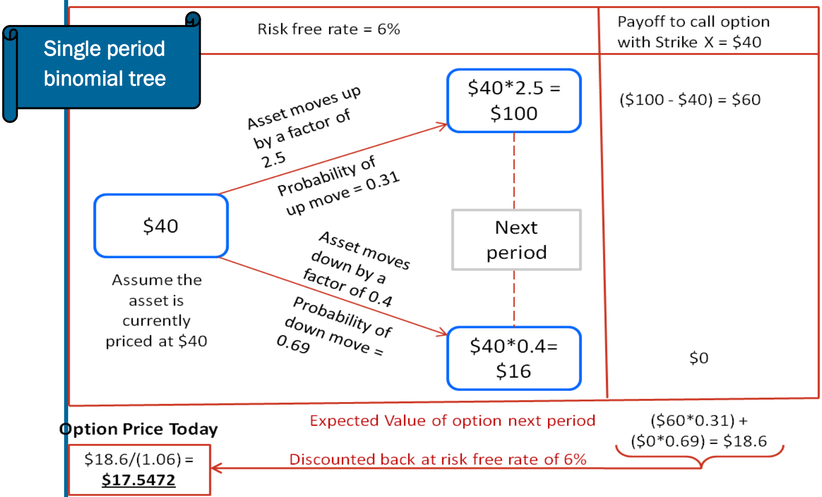
Hence we can clearly see that delta for calls is positive and delta for puts is negative. We can also see that delta by definition is the instantaneous rate of change of option value with respect to stock price. The linear approximation for delta as shown in our measure of the slope above only holds for small changes in the value of asset price.

Further, Delta tends to 0 for deep out of the money calls and puts. Delta tends to 1 for deep in the money calls and –1 for deep in the money puts.

Delta versus passing time

For a constant stock price, as time passes	**Calls**	**Puts**
Out of the money	Delta tends to 0	Delta tends to 0
In the money	Delta tends to 1	Delta tends to –1

Delta Hedging



Earlier we'd seen this option pricing model. What if the market didn't price the option at $17.55 like the model dictates? Let's say the option is overpriced by the market and is trading at ***$19.00***. There's an arbitrage possibility possible here!

Here's how it works—

(Assume a risk free borrowing rate of 6%)

a) Short 100 options and purchase shares as shown by the following equation-
Delta = (C+ - C-) / (S+ - S-) = ($60—$0)/($100—$16) = 0.7143 shares per option or 71.43 shares. This is called the "hedged delta-neutral portfolio" and the ratio is called the "hedge ratio"

c) This portfolio today costs (71.43x$40) - ($19x100) = $957.20. We'll finance this by borrowing at the risk free rate. In one year we'll owe $957.2(1.06) = $1014.63

d) Over the next period the value of portfolio in the 2 cases is—

Uptick: (71.43*$100) - (100*$60) = $1143

Downtick: (71.43*$16) - (100*$0) = $1143

The dollar return in either case is $1143—1014.63 = $128.37

We are making money on the delta here! This means that we make money from a non-homogeneous expectation of the price of the asset over the next period and hence the payout to the option at maturity.

What happens if the option was underpriced?

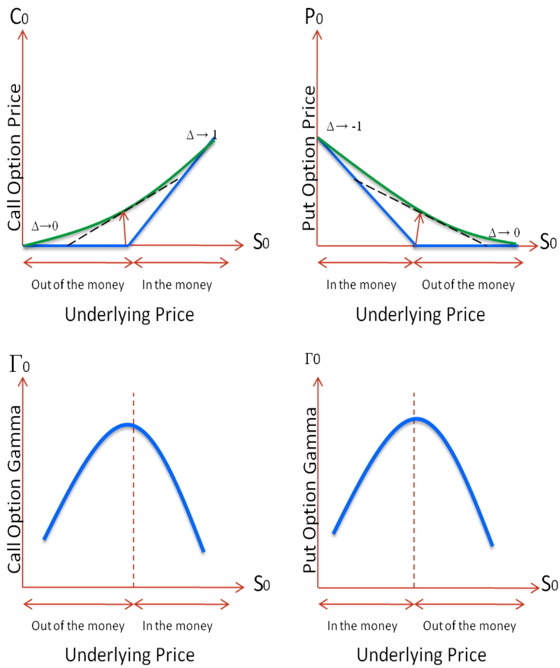
In this case the hedge portfolio will consist of a long position in the option and a short position in the underlying!

Dynamically managing the delta hedge

As the price of the underlying asset changes,
The delta of the option changes,
Hence the hedge ratio of the number of options required per share to maintain the delta-neutrality of the portfolio changes,
So we'll have to actively manage the portfolio by trading for the required number of options / stock required to maintain the portfolio.

Note: Maintaining the delta neutrality comes with its share of transaction costs.

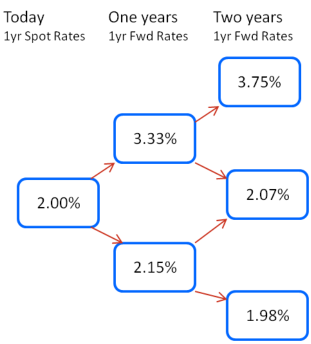
Use of higher order Greeks to study the hedge position

Gamma(Γ)	
Formulae	$\Gamma(calls\ and\ puts) = \dfrac{\partial^2 C_0}{\partial S_0{}^2} = \dfrac{N'(d1)}{S_0 \sigma T}$
What it means	Gamma is a second order Greek. It is the first partial differential of delta with respect to asset price or the second partial differential of option value with respect to asset price. Simply gamma is the "rate of change of delta". Gamma is high when a call or put option is most sensitive to changes in the underlying stock price. This happens when the options are at the money! Gamma is low when delta is insensitive to asset price changes. This happens when the option is deep in the money or deep out of the money!
Plots	

Interpretation	For options that are at the money, Gamma is high, Hence, in response to a small change in asset price, the option price will change greatly. This means that the dynamic hedge will perform poorly and frequent rebalancing is necessary. For deep in the money or out of the money options, the change in delta in response to a unit change in asset price is insignificant, and hence rebalancing of the hedge portfolio can be performed less frequently, and hence transaction costs will be lower.
Example	Gamma for an option is 0.06. How much will a call option price change for a \$2 rise in asset price? (Delta was initially 0.43) **Answer:** For a \$2 change in asset price delta will change by 2*0.06 = 0.12 As asset price increases, delta for a call option would have increased by 0.12. Hence the new delta is 0.55. The value of the call option value will have hence increased by \$2*0.55 = \$1.1

DIY Time!

1) The tree below is how you believe interest rates are expected to move over the next two periods. Your task is to price up an interest rate floor with a strike set at 3% that you'd like to purchase to set the minimum on interest payments for a \$10,000,000 notional loan you've made that has 2 years to run to maturity.

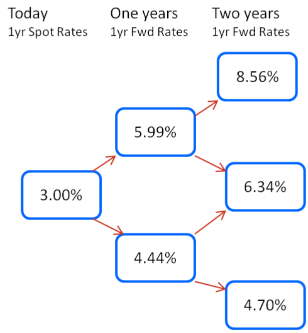


2) **Delta Hedging**

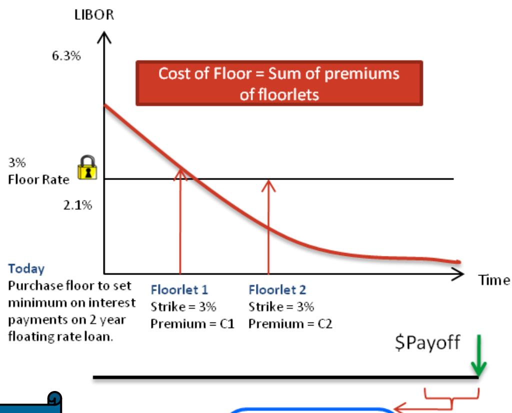
Suggest how to construct a delta neutral portfolio if we were to capitalize on an underpriced –0.54 delta put option in a trade involving a lot of 100 puts.

DIY Time!

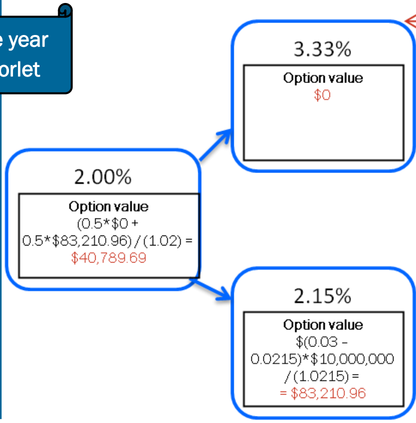
3) Evaluate an American put option with 2 years to expiration and a ***strike price of $102.50***. The underlying bond is a $100 par value, 8% annual coupon bond with three years to maturity. Compute the value of the option today.



Answers



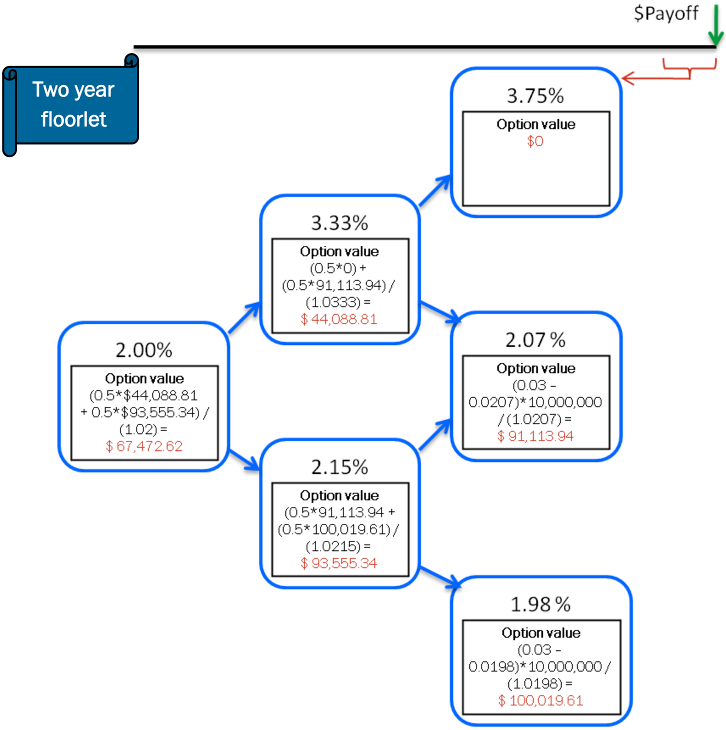
One year floorlet



<u>Value of the 2 year floor</u>

= Value of 1 year floorlet
+ Value of 2 year floorlet

= $40,789.69+
$67,472.62
= $117,262.31



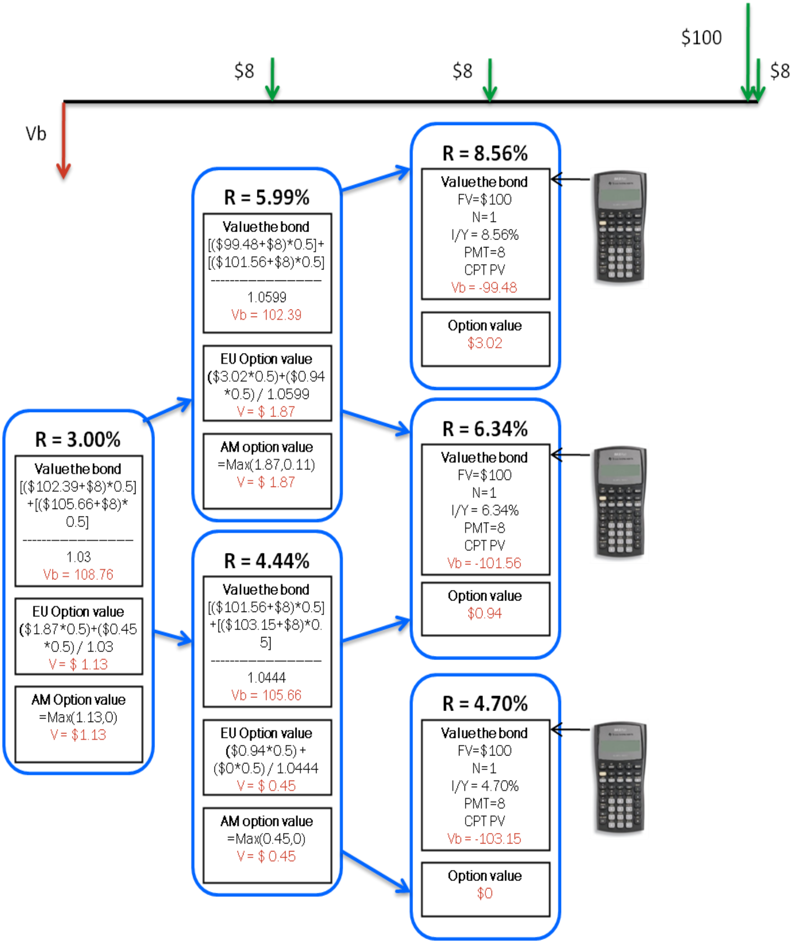
2) If the put is underpriced, we will buy 100 puts. Now these puts in isolation will payoff if price falls below the strike and will not pay when asset price rises. If we were to limit the payout were asset prices to trend downward, and transfer some of this benefit over to asset price upswing we will have to go LONG on the asset!

Hence the hedge portfolio will consist of a long put and a long asset position.

The number of shares of the underlying we will need to buy is 100*0.54 = 54 shares!

Answers

3) Answer: $1.13



Notes

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